Delta sets for divisors supported in two points

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Goal

Goal :

- Give motivation for the delta sets.
- Improve the bounds for the minimum distance of AG codes.

Code

- $\mathbb{F}$: finite field
- $\mathbb{F}$-linear code $\mathcal{C}$ of length n : linear subspace of $\mathbb{F}^n$
- Hamming distance of $x, y \in \mathbb{F}^n$: $d(x, y) = |\{i : x_i \neq y_i\}|$
- Minimum distance of $\mathcal{C}$:

$$\begin{aligned} d(\mathcal{C}) &= \min \{d(x, y) : x, y \in \mathcal{C}, x \neq y\} \\ &= \min \{d(x, 0) : x \in \mathcal{C}, x \neq 0\}. \end{aligned}$$

- $x * y$: coordinate-wise product of x and y

Algebraic geometric code

- $X/\mathbb{F}$: algebraic curve (absolutely irreducible, smooth, projective) of genus g over finite field $\mathbb{F}$
- $\mathbb{F}(X)$: function field of $X/\mathbb{F}$
- $L(A) := \{f \in \mathbb{F}(X)\backslash\{0\} \ : \ (f) + A \geq 0 \ \} \cup \{0\}$
- $\Omega(A) := \{\omega \in \Omega(X)\backslash\{0\} \ : \ (\omega) \geq A \ \} \cup \{0\}$
- K : canonical divisor
- $P_1, \ldots, P_n$: n distinct rational points on X
- $D = P_1 + \cdots + P_n$
- G : divisor, $\mathrm{supp}(G) \cap \mathrm{supp}(D) = \emptyset$

Example

$(f) = 5P - 3Q$

- f has zeros of order 5 at P
- f has poles of order 3 at Q
- $\deg(\ (f)\ ) = 5 - 3 = 2$

If $A = 4P - 2Q$ then $L(A) := \{f \in \mathbb{F}(X)\backslash\{0\}\ :\ (f) + A \geq 0\ \} \cup \{0\}$ consists of all elements $f \in \mathbb{F}(X)$ such that

- f has zeros of order at least 2 at Q
- f may have a pole at P of order at most 4

Algebraic geometric code :

$$\alpha_L : L(G) \longrightarrow \mathbb{F}^n, \quad f \mapsto (f(P_1), \ldots, f(P_n)),$$
$$\alpha_\Omega : \Omega(G - D) \longrightarrow \mathbb{F}^n, \quad \omega \mapsto (\mathrm{res}_{P_1}(\omega), \ldots, \mathrm{res}_{P_n}(\omega))$$
$$\mathrm{Im}(\alpha_L) = C_L(D, G)$$
$$\mathrm{Im}(\alpha_\Omega) = C_\Omega(D, G)$$

Algebraic geometric code $C_L(D, G)$:

$$C_L(D, G) := \{(f(P_1), \ldots, f(P_n)) \mid f \in L(G)\} \subseteq \mathbb{F}^n$$

Algebraic geometric code $C_\Omega(D, G)$:

$$C_\Omega(D, G) := \{(\mathrm{res}_{P_1}(\omega), \ldots, \mathrm{res}_{P_n}(\omega) \mid \omega \in \Omega(G - D)\} \subseteq \mathbb{F}^n$$

Algebraic geometric code

- $C_L(D, G)$ and $C_\Omega(D, G)$ are dual to each other
- $C_L(D, G)^\perp = C_\Omega(D, G) = C_L(D, K + D - G)$
- For $a, b \in \mathbb{Z}$ and $P, Q \in X(\mathbb{F})$,

$$G = aP \quad : \quad \text{one-point code}$$
$$G = aP + bQ \quad : \quad \text{two-point code}$$

Order bounds of the minimum distance

- Hamming distance between two nonempty subsets $X, Y \subset \mathbb{F}^n$:

$$\text{minimum of } \{d(x,y) : x \in X, y \in Y\}$$

- For a proper subcode $\mathcal{C}' \subset \mathcal{C}$, the minimum distance of the collection of cosets $\mathcal{C}/\mathcal{C}'$ is

$$\begin{aligned} d(\mathcal{C}/\mathcal{C}') &= \min\{d(x+\mathcal{C}', y+\mathcal{C}') : x, y \in \mathcal{C}, x-y \notin \mathcal{C}'\} \\ &= \min\{d(x,0) : x \in \mathcal{C}, x \notin \mathcal{C}'\}. \end{aligned}$$

Theorem (Coset bound)
Let $\mathcal{C}/\mathcal{C}_1$ be an extension of $\mathbb{F}$-linear codes with corresponding extension of dual codes $\mathcal{D}_1/\mathcal{D}$ such that $\dim \mathcal{C}/\mathcal{C}_1 = \dim \mathcal{D}_1/\mathcal{D} = 1$.
If there exist vectors $a_1, \ldots, a_w$ and $b_1, \ldots, b_w$ such that

$$\begin{cases} a_i * b_j \in \mathcal{D} & \text{for } i+j \leq w, \\ a_i * b_j \in \mathcal{D}_1 \backslash \mathcal{D} & \text{for } i+j = w+1, \end{cases}$$

then $d(\mathcal{C}/\mathcal{C}_1) \geq w$.

Corollary Let $\mathcal{C}/\mathcal{C}'$ be an extension of $\mathbb{F}$-linear codes of length n. For $\mathcal{C} \supset \mathcal{C}'' \supset \mathcal{C}'$,

$$d(\mathcal{C}/\mathcal{C}') = \min\{d(\mathcal{C}/\mathcal{C}''), d(\mathcal{C}''/\mathcal{C}')\}.$$

$$d(\mathcal{C}) = \min\{d(\mathcal{C}/\mathcal{C}'), d(\mathcal{C}')\}.$$

Hermitian curve $X/\mathbb{F}_{q^2} : y^q + y = x^{q+1}$

- Number of rational points : $q^3 + 1$
- Genus : $g = q(q-1)/2$
- P_∞ : the point at infinity of X
- P_0 : the point $(0, 0)$
- $K = (q-2)H$, where $H \sim (q+1)P_\infty \sim (q+1)P_0$

Example of one-point Hermitian code :

$X : y^4 + y = x^5$ over $\mathbb{F}_{16}$

Number of rational points $= 65$: $P_1, \ldots, P_{64}, P_\infty$

$g = 6$

Canonical divisor : $K = 10P_\infty$

For $f \in \mathbb{F}_{16}(X)\backslash\{0\}$,

$(f)_\infty$: the pole divisor of f

$(f)_0$: the zero divisor of f

$(f) = (f)_0 - (f)_\infty$

Weierstrass semigroup of the point P_∞:

$$H(P_\infty) = \{n \in \mathbb{N}_0 : \exists\ f \in \mathbb{F}_{16}(X) \ \text{ with } \ (f)_\infty = nP_\infty\}$$

$(x)_\infty = 4P_\infty$

$(y)_\infty = 5P_\infty$

Gap numbers : 1,2,3,6,7,11

$$H(P_\infty) = \langle 4, 5\rangle = \{0,\ 4,\ 5,\ 8,\ 9,\ 10,\ 12,\ 13,\ 14,\ 15,\ 16, \ldots\}$$

Question :

For $G = 17P_\infty$ and $D = P_1 + \cdots + P_{64}$,

$d(C_\Omega(D, G))= ?$

$$\mathcal{C}/\mathcal{C}_1 = C_\Omega(D, 17P_\infty)/C_\Omega(D, 18P_\infty) \longleftrightarrow \mathcal{D}_1/\mathcal{D} = C_L(D, 18P_\infty)/C_L(D, 17P_\infty)$$

If there exist vectors $a_1, \ldots, a_w$ and $b_1, \ldots, b_w$ such that

$$\begin{cases} a_i * b_j \in \mathcal{D} = C_L(D, 17P_\infty) & \text{for } i + j \leq w, \\ a_i * b_j \in \mathcal{D}_1 \backslash \mathcal{D} = C_L(D, 18P_\infty) \backslash C_L(D, 17P_\infty) & \text{for } i + j = w + 1, \end{cases}$$

then $d(\mathcal{C}/\mathcal{C}_1) \geq w$.

- In other words, if there exist rational functions $f_1,\ f_2, \ldots, f_w$ and $g_1,\ g_2, \ldots, g_w$ such that

$$\begin{cases} f_i g_j \in L(17P_\infty) & \text{for } i+j \leq w, \\ f_i g_j \in L(18P_\infty) \backslash L(17P_\infty) & \text{for } i+j = w+1, \end{cases}$$

  then $d(\mathcal{C}/\mathcal{C}_1) \geq w$.

- $f_i g_j \in L(18P_\infty) \backslash L(17P_\infty)$ means that $f_i g_j$ has a pole only at P_∞ with exact pole order 18, that is, $(f_i g_j)_\infty = 18P_\infty$.

Hermitian curve

			f_1	f_2	f_3	f_4	f_5	f_6		f_7	f_8				f_9
			1	x	y	x^2	xy	y^2	x^3	x^2y	xy^2	y^3	x^4	x^3y	x^2y^2
			0	4	5	8	9	10	12	13	14	15	16	17	18
g_1	1	0													18
g_2	x	4									18				
g_3	y	5								18					
g_4	x^2	8						18							
g_5	xy	9					18								
g_6	y^2	10				18									
	x^3	12													
g_7	x^2y	13			18										
g_8	xy^2	14		18											
	y^3	15													
	x^4	16													
	x^3y	17													
g_9	x^2y^2	18	18												

It follows from the figure that

$$\begin{cases} f_i g_j \in L(17P_\infty) & \text{for } i+j \leq 9, \\ f_i g_j \in L(18P_\infty) \backslash L(17P_\infty) & \text{for } i+j = 10. \end{cases}$$

Thus $d(\mathcal{C}/\mathcal{C}_1) \geq 9$.

✓				✓	✓			✓	✓	✓			✓	✓				✓
0	~~1~~	~~2~~	~~3~~	4	5	~~6~~	~~7~~	8	9	10	~~11~~	12	13	14	15	16	17	18
18	17	16	15	14	13	12	~~11~~	10	9	8	~~7~~	~~6~~	5	4	~~3~~	~~2~~	~~1~~	0

Applying the same method to $C_\Omega(D, 18P_\infty)/C_\Omega(D, 19P_\infty)$ we have

$$d(C_\Omega(D, 18P_\infty)/C_\Omega(D, 19P_\infty)) \geq 8.$$

≥ 9	$C_\Omega(D, 17P_\infty) \setminus C_\Omega(D, 18P_\infty)$
≥ 8	$C_\Omega(D, 18P_\infty) \setminus C_\Omega(D, 19P_\infty)$
≥ 9	$C_\Omega(D, 19P_\infty) \setminus C_\Omega(D, 20P_\infty)$
≥ 10	$C_\Omega(D, 20P_\infty) \setminus C_\Omega(D, 21P_\infty)$
	$\vdots$

By taking the minimum of the weights of the codewords, we have

$$d(C_\Omega(D, G)) \geq 8.$$

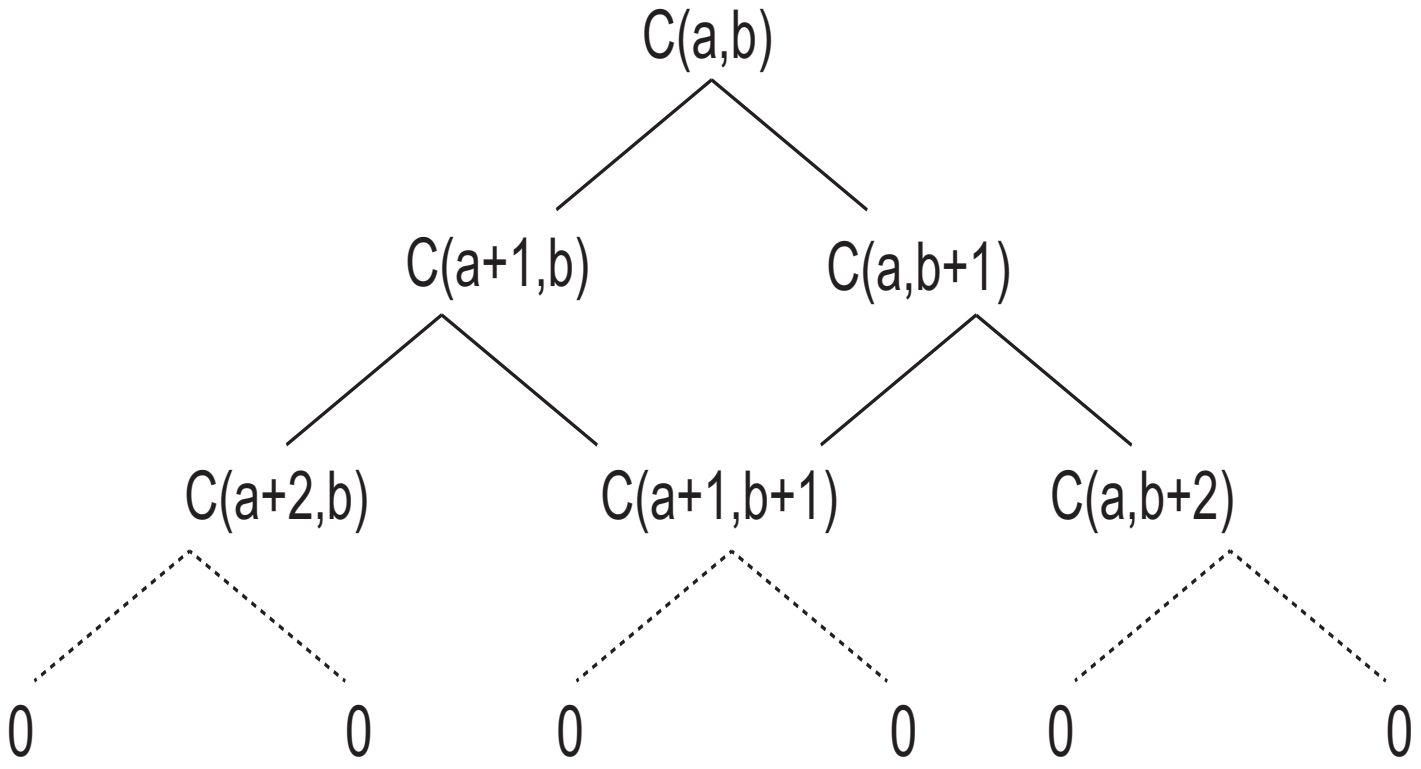
Notation : $C_\Omega(D, G) = C(a, b)$, where $G = aP_\infty + bP_0$

In two-point Hermian code with $q = 4$:

If $G = 20P_\infty + P_0$ then what is $d(C(20P_\infty + P_0))$?

$C(20P_\infty + P_0) \longrightarrow C(21P_\infty + P_0)$

$C(20P_\infty + P_0) \longrightarrow C(20P_\infty + 2P_0)$


C(a,b)
C(a+1,b)
C(a,b+1)
C(a+2,b)
C(a+1,b+1)
C(a,b+2)
0
0
0
0
0
0

Order bounds for the minimum distance

- For the Feng-Rao bound the filtration is determined by the choice of a point P and takes the form

$$C_\Omega(D, G) \supset C_\Omega(D, G + P) \supset C_\Omega(D, G + 2P) \supset \cdots \supset \{0\}$$

- Beelen allows the addition of different points at different steps in the filtration.

$$d(C_L(D,G)) = \min\{\deg A : 0 \leq A \leq D \mid L(G-D+A) \neq L(G-D)\}$$

$$= \min\{\deg A : 0 \leq A \leq D \mid L(A-C) \neq L(-C)\}, \quad \text{for } C = D-G$$

$$d(C_\Omega(D,G)) = \min\{\deg A : 0 \leq A \leq D \mid \Omega(G-A) \neq \Omega(G)\}$$

$$= \min\{\deg A : 0 \leq A \leq D \mid L(K-G+A) \neq L(K-G)\}$$

$$= \min\{\deg A : 0 \leq A \leq D \mid L(A-C) \neq L(-C)\}, \quad \text{for } C = G-K$$

Notation

$\mathcal{C}/\mathcal{C}_1 = C_L(D, G)/C_L(D, G - P)$

$\mathcal{D}_1/\mathcal{D} = C_\Omega(D, G - P)/C_\Omega(D, G)$

$$d(\mathcal{C}/\mathcal{C}_1) = \min\{\deg A : 0 \leq A \leq D \mid L(A - C) \neq L(A - C - P)\},$$
$$\text{for } C = D - G$$

$$d(\mathcal{D}_1/\mathcal{D}) = \min\{\deg A : 0 \leq A \leq D \mid L(A - C) \neq L(A - C - P)\},$$
$$\text{for } C = G - K - P$$

$P \notin D$ and $0 \leq A \leq D \Longrightarrow L(A) \neq L(A - P)$

This motivates the following definition :

$$\Gamma_P(C) = \{A : L(A) \neq L(A - P) \wedge L(A - C) \neq L(A - C - P)\}$$

Definition :

$\Gamma_P = \{A : L(A) \neq L(A - P)\}$

$\Gamma_P(C) = \{A \in \Gamma_P : A - C \in \Gamma_P\}$

$\gamma_P(C) = \min\{\deg A : A \in \Gamma_P(C)\}$

$\Delta_P(C) = \{A \in \Gamma_P : A - C \notin \Gamma_P\}$

Theorem :

For a rational point $P \notin D$,

$$d(C_L(D, G)/C_L(D, G - P)) \geq \gamma_P(D - G).$$

$$d(C_\Omega(D, G)/C_\Omega(D, G + P)) \geq \gamma_P(G - K).$$

All divisors of sufficiently large degree belong to $\Gamma_P(C)$ while the degree of a divisor in $\Delta_P(C)$ is bounded.

$A \in \Delta_P(C) \Leftrightarrow K + C + P - A \in \Delta_P(C)$.

For $A \in \Delta_P(C)$, $0 \leq \deg A \leq \deg C + 2g - 1$.

Theorem (Coset bound for divisors) :

Let $\{A_1 \leq A_2 \leq \cdots \leq A_w\} \subset \Delta_P(C)$ be a sequence of divisors with $A_{i+1} \geq A_i + P$, for $i = 1, \ldots, w-1$. Then $\deg A \geq w$, for every divisor $A \in \Gamma_P(C)$ with support disjoint from $A_w - A_1$

Sketch of Proof :

We obtain two sequences of subspaces.

$$L(A_w) \supsetneq L(A_w - P) \supseteq L(A_{w-1}) \supsetneq L(A_{w-1} - P) \supseteq \cdots$$
$$\cdots \supseteq L(A_2) \supsetneq L(A_2 - P) \supseteq L(A_1) \supsetneq L(A_1 - P).$$

$$\Omega(A_w - C) \subsetneq \Omega(A_w - C - P) \subseteq \Omega(A_{w-1} - C) \subsetneq \Omega(A_{w-1} - C - P) \subseteq \cdots$$
$$\cdots \subset \Omega(A_2 - C) \subsetneq \Omega(A_2 - C - P) \subseteq \Omega(A_1 - C) \subsetneq \Omega(A_1 - C - P).$$

For $i = 1, 2, \ldots, w$, choose

$$f_i \in L(A_i) \backslash L(A_i - P) \quad \text{and} \quad \eta_i \in \Omega(A_i - C - P) \backslash \Omega(A_i - C).$$

For a given divisor B,

$$\Delta_P(B, C) = \{B + iP : i \in \mathbb{Z}\} \cap \Delta_P(C).$$

Corollary :

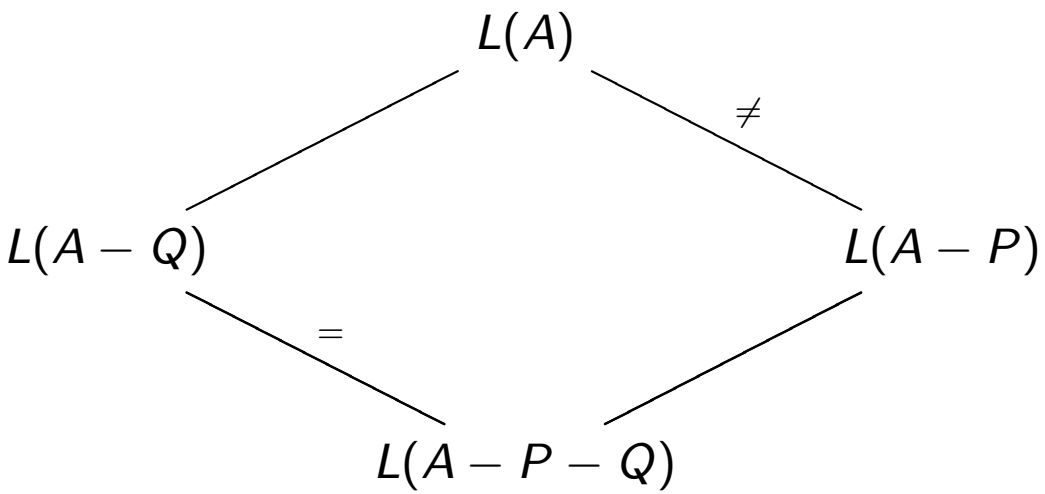
For any choice of divisor B,

$$\gamma_P(C) \geq \#\Delta_P(B, C).$$

Lemma :

For distinct points P and Q, $\Delta_P(Q) = \Delta_Q(P)$.

Sketch of Proof :



Definition :

Let $D(P,Q) = \Delta_P(Q) = \Delta_Q(P)$. A divisor A is called a discrepancy for the points P and Q if $A \in D(P,Q)$.

Theorem :

$$\dim L(B + aP + bQ) = \#\{B + iP + jQ \in D(P,Q) : i \leq a \text{ and } j \leq b\}.$$

Definition :

$$D_B(P,Q) = D(P,Q) \cap \{B + iP + jQ : i,j \in \mathbb{Z}\}.$$

Theorem :

For distinct points P and Q, and for a given divisor B, there exist functions $\sigma = \sigma_B$ and $\tau = \tau_B$ such that

$$D_B(P, Q) = \{B + \tau(j)P + jQ : j \in \mathbb{Z}\} = \{B + iP + \sigma(i)Q : i \in \mathbb{Z}\}.$$

$\sigma = \sigma_B$, $\tau = \tau_B$: mutual inverses and describe permutations of the integers. For m minimal such that $mP \sim mQ$, the functions σ, τ are determined by their images on a full set of representatives for $\mathbb{Z}/m\mathbb{Z}$. and $D_B(P, Q)$ consists of m distinct divisor classes.

Hermitian Curve over $\mathbb{F}_{q^2}$:

$$X/\mathbb{F}_{q^2} : y^q + y = x^{q+1}$$

- For $P, Q \in X(\mathbb{F}_{q^2})$, $mP \sim mQ$ for $m = q + 1$
- The m inequivalent divisor classes in $D_0(P, Q)$ with support in P and Q are represented by the divisors

$$dH - dP - dQ, \quad \text{for } d = 0, 1, \ldots, q.$$

- $D_0(P, Q) = \{d(qQ - P) + j(q + 1)(P - Q) : d = 0, 1, \ldots, q, j \in \mathbb{Z}\}$.

Suzuki Curve over $\mathbb{F}_q$:

$$y^q + y = x^{q_0}(x^q + x), \qquad q_0 = 2^n, \qquad q = 2q_0^2 = 2^{2n+1}$$

- Number of rational points : $q^2 + 1$
- Genus : $g = q_0(q-1)$
- P_∞ : the unique pole of x
- P_0 : the unique zero of both x and y
- Semigroup of Weierstrass nongaps at P_∞
  $= \langle q, q+q_0, q+2q_0, q+2q_0+1 = m \rangle$
- $K \sim 2(q_0-1)H$, where $H \sim (q+2q_0+1)P_\infty \sim (q+2q_0+1)P_0$

Theorem :

The m inequivalent divisor classes in $D_0(P,Q)$ are represented by

$$\begin{aligned} &iD_0 + jD_2, \quad \text{for } 0 \le i,j \le q_0, \text{ and} \\ &D_1 + i'D_0 + j'D_2, \quad \text{for } 0 \le i',j' \le q_0 - 1. \end{aligned}$$

The given representatives correspond one-to-one to the m divisors

$$D(a,b) = (a+q_0)H - ((a+q_0)(q_0+1)+bq_0)P - ((a+q_0)(q_0+1)-bq_0)Q,$$

for $|a| + |b| \le q_0$.

Suzuki curve over $\mathbb{F}_8$:

$X/\mathbb{F}_8 : y^8 + y = x^{10} + x^3$

$g = 14, N = 65,, m = 13$

The m inequivalent divisor classes in $D(P, Q)$ with support in P and Q are represented by the divisors

$$\begin{array}{ccccc} (0,0) & \cdot & (-5,12) & \cdot & (-10,24) \\ \cdot & (-3,10) & \cdot & (-8,22) & \cdot \\ (-1,8) & \cdot & (-6,20) & \cdot & (-11,32) \\ \cdot & (-4,18) & \cdot & (-9,30) & \cdot \\ (-2,16) & \cdot & (-7,28) & \cdot & (-12,40) \end{array}$$

Theorem :

Let $A = kP + \ell Q$ and $C = iP + jQ$. Let

$$k^- + \ell = d_Q(\ell - j) + i + j, \qquad d^- = d_P(k^-) - d_Q(\ell - j).$$
$$k^+ + \ell = d_Q(\ell), \qquad d^+ = d_Q(\ell) - d_P(k^+ - i).$$

Then

$$(1)\ \ A \notin \Delta_P(C) \ \wedge\ A - Q \in \Delta_P(C) \ \Leftrightarrow\ k = k^- \ \wedge\ i + j > d^-.$$
$$(2)\ \ A \in \Delta_P(C) \ \wedge\ A - Q \notin \Delta_P(C) \ \Leftrightarrow\ k = k^+ \ \wedge\ i + j > d^+.$$

$kP + \ell Q \in \Gamma_P \Longleftrightarrow k + \ell \geq d_P(k)$
$kP + \ell Q \in \Gamma_Q \Longleftrightarrow k + \ell \geq d_Q(\ell)$
For $mP \sim mQ$, the functions d_P and d_Q are defined modulo m.
$d_p(k) = k + \sigma(k)$ and $d_Q(\ell) = \ell + \tau(\ell)$

Example :

For the Suzuki curve over $\mathbb{F}_{32}$, let $C = 55P + 31Q$. The full grid contains a unique maximal sequence of length 90 in row $\ell = -5$,

$$\begin{aligned}\Delta_P(55P + 31Q) \supseteq &\{A_1 = 36P - 5Q, \ldots, A_{45} = 163P - 5Q\} \\ &\cup \{A_{46} = 180P - 5Q, \ldots, A_{90} = 307P - 5Q\}\end{aligned}$$

THANK YOU!